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THE DEVELOPMENT OF STATISTICAL METHODS FOR QUALITY CONTROL AND SURVEILLANCE TESTING

SUBJECTIVE TESTING AND QUALITY EVALUATION

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THE MODIFIED TRIANGLE TEST

by

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I. SUMMARY.

The triangle test has been used very frequently in sensory difference testing. Attempts have been made to augment the basic information from triangle tests with scores on degree of difference between the sample selected as the variant and the remaining two. In this paper a mathematical model has been developed to permit formal utilization of the degree of difference scores.

It is assumed that a stimulus-response scale exists and that two standard samples and one variant sample evoke responses x_1, x_2 and y on this scale. These responses are taken to have independent normal distributions with variances σ^2 and means zero for the x-variates and mean μ for the y-variate. Conditional distributions for degree of difference scores are obtained, the likelihood function for N independent trials of the triangle test is developed, and a procedure for testing that $\mu = 0$ is given.

The procedure considered is designated as the modified triangle test and illustrations of its use are shown.

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II. INTRODUCTION.

Sensory difference tests are used frequently in many situations involving the subjective responses of individuals. These tests may be used for the maintenance of quality, the detection of adulterants, the substitution of ingredients, the screening of product variants, the measurement of sensitivities of individuals, and the selection of test panels. The triangle test is perhaps the most frequently used of the simple sensory difference tests and is the subject of this paper.

There has been a certain amount of confusion in the literature on the choice of an appropriate sensory difference test. Experimental comparisons have been made, for example, by Liebmann and Panettiere [1957]. Mathematical and statistical comparisons have been made by Hopkins and Gridgeman [1955], Radkins [1957], Ura [1960] and Bradley [1958, 1963]. These comparisons suggest the use of the triangle test in most situations where a simple procedure is required.

Efforts have been made to modify the traingle test in order to obtain improved performance in the detection of differences. Much of this work appears to be unpublished but an example is given by Mahoney, Stier and Crosby [1957]. The basic triangle test can be discussed in terms of discrimination between a standard product and a variant from that standard. A group of N respondents are each presented a traingle test. The presentation involves two samples of the standard and one of the variant, all unidentified, and it is the task of the respondent to select the variant sample from the three samples on the basis of the sensory characteristic under study. In the modifications of the triangle test considered, an additional task is imposed on the respondent. For the modified test, the respondent not only must select the variant sample but must also "score" the "degree of difference" present. This is usually done on the basis of a crude scale implying that the difference is "slight," "moderate," "easily detectable," "extreme," etc. In past use of the modified triangle test, it appears that the scores provided have been used largely as a qualitative check on the test rather than for specific analysis.

It is the purpose of this paper to provide a method of analysis for modified triangle tests. In providing this method, certain assumptions are made that require further study but it is believed that a useful technique has been developed to improve sensory difference testing.

III. FORMULATION OF THE PROBLEM.

A basis for the analysis of modified triangle tests has been provided by Bradley and Ura in the papers cited. Furthermore, that work indicated that the resulting analysis may not be too sensitive to the assumptions made.

Following Bradley [1963] we suppose a conceptual, sensory-difference, stimulus-response scale for sensory sensations of respondents in difference testing. Assume that responses to the standard samples are normally distributed with mean zero (the origin of such a scale is arbitrary) and variance σ^2. The responses to the variant are likewise assumed to be normally distributed with variance σ^2 but with mean μ. All these stimulus responses are taken to be stochastically independent. Let the two responses to the standard be x_1 and x_2 and the response to the variant be y. Whether or not the respondent correctly selects the odd or variant sample from the three is assumed to depend on the configuration of the responses x_1, x_2 and y. Twelve possible configurations may be obtained and are designated by $C_1,\ldots,C_{12}$ as follows:

$C_1: x_1 < x_2 < y, \quad x_2 - x_1 < y - x_2;$

$C_2: x_2 < x_1 < y, \quad x_1 - x_2 < y - x_1;$

$C_3: y < x_1 < x_2, \quad x_2 - x_1 < x_1 - y;$

$C_4: y < x_2 < x_1, \quad x_1 - x_2 < x_2 - y;$

$C_5: x_1 < x_2 < y, \quad y - x_2 < x_2 - x_1;$

$C_6: x_2 < x_1 < y, \quad y - x_1 < x_1 - x_2;$

$C_7: y < x_1 < x_2, \quad x_1 - y < x_2 - x_1;$

$C_8: y < x_2 < x_1, \quad x_2 - y < x_1 - x_2;$

$C_9: x_1 < y < x_2, \quad x_2 - y < y - x_1;$

$C_{10}: x_2 < y < x_1, \quad x_1 - y < y - x_2;$

$C_{11}: x_1 < y < x_2, \quad y - x_1 < x_2 - y;$

$C_{12}: x_2 < y < x_1, \quad y - x_2 < x_1 - y.$

The first four of these configurations lead to correct selection on the triangle test and the score on degree of difference is taken to approximate to $R = |y - \frac{1}{2}(x_1 + x_2)|$. The last eight of these configurations lead to incorrect selection and the score on degree of difference measures either $W = |x_1 - \frac{1}{2}(y + x_2)|$ or $W = |x_2 - \frac{1}{2}(y + x_1)|$. In the following work it is assumed that R and W have distributions based on the indicated functions of normally distributed variates even though observed values of R and W are imperfectly measured on discrete scoring scales.

At this stage note that data from N modified triangle tests will appear as scores, $R_1,\ldots,R_m$, $W_1,\ldots,W_n$, $m + n = N$, in some order with m and n themselves random variables. On the basis of such data, a test of significance is developed. The null hypothesis is that $\mu = 0$, there is no difference between standard samples and variant samples on the basis of the sensory attributes under study. The alternative hypothesis is that $\mu \neq 0$. The hypothesis that $\mu = 0$ is equivalent, of course, to the more standard statement that the probability of correct selection of the variant sample is 1/3, a selection based on chance alone. If $p_\triangle$ is the probability that correct selection be made, $p_\triangle$ is related to μ/σ as given by Bradley [1963]:

$$p_{\triangle} = \frac{e^{-\mu^2/3\sigma^2}}{\sqrt{\pi}} \sum_{i=0}^{\infty} \frac{(\mu^2/3\sigma^2)^i}{\Gamma(i+\frac{1}{2})} \int_3^{\infty} \frac{u^{i-\frac{1}{2}}}{(1+u)^{i+1}}\, du. \tag{1}$$

(Bradley and Ura have given tables associating values of $p_{\triangle}$ with values of μ/σ The next section deals with the problem of obtaining distributions of R and W based on the assumptions made.

IV. DISTRIBUTIONS OF DIFFERENCE SCORES.

The joint likelihood function of $R_1, \ldots, R_m$, $W_1, \ldots, W_n$ may be built up from the conditional distributions of R and W given the appropriate configurations from $C_1, \ldots, C_{12}$. Configuration C_1 leads to a value of R and we develop now the conditional probability density function of R given C_1 in some detail.

The conditional joint probability density function of x_1, x_2 and y given C_1, on the assumption of the normal distributions and independence, is

$$f(x_1, x_2, y|C_1) = \frac{(2\pi\sigma^2)^{-3/2}}{P(C_1)} e^{-\{x_1^2 + x_2^2 + (y-\mu)^2\}/2\sigma^2}, \tag{2}$$

$$x_1 < x_2 < y, \quad x_2 - x_1 < y - x_2, \quad -\infty \leq x_1, x_2, y \leq \infty,$$

where $P(C_1)$ is the probability of occurrence of configuration C_1. It is clear that (2) follows from the joint unconditional distribution of x_1, x_2 and y which differs from (2) only in the omission of the division by $P(C_1)$ and the first two restricting inequalities following (2). An orthogonal transformation to new variables r, s and t is made with $r = (2y - x_1 - x_2)/\sqrt{6}$, $s = (y + x_1 + x_2)/\sqrt{3}$, and $t = (x_2 - x_1)/\sqrt{2}$. Then the joint probability density for the new variables is

$$f(r,s,t|C_1) = \frac{(2\pi\sigma^2)^{-3/2}}{P(C_1)} e^{-\{r^2 + s^2 + t^2 - (4\mu r/\sqrt{6}) - (2\mu s/\sqrt{3}) + \mu^2\}/2\sigma^2}, \quad (3)$$

$$0 < t < (r/\sqrt{3}) \leq \infty , \quad -\infty \leq s \leq \infty.$$

It is seen that r is related to the required R and integration with respect to s and t leads to the marginal conditional distribution of r,

$$f(r|C_1) = \frac{(2\pi\sigma^2)^{-\frac{1}{2}}}{P(C_1)} I(r/\sigma\sqrt{3}) e^{-\{r^2 - (4\mu r/\sqrt{6}) + (2\mu^2/3)\}/2\sigma^2}, \quad (4)$$

$0 < r \leq \infty$. In (4) the function I is the incomplete standard normal integral,

$$I(a) = (2\pi)^{-\frac{1}{2}} \int_0^a e^{-u^2/2} du. \quad (5)$$

Since $R = |y - \frac{1}{2}(x_1 + x_2)|$ and $r > 0$, $R = \sqrt{6}\, r/2$ and the required conditional probability density function for R is

$$f(R|C_1) = \frac{(3\pi\sigma^2)^{-\frac{1}{2}}}{P(C_1)} e^{-(R - \mu)^2/3\sigma^2} I(\sqrt{2} R/3\sigma) , \quad (6)$$

$0 \leq R \leq \infty$.

It follows from symmetry that $f(R|C_2)$ has the same form as (6) and indeed $P(C_2) = P(C_1)$. $f(R|C_3)$ and $f(R|C_4)$ have the form of (6) but with μ replaced by $-\mu$. Accordingly, we may compound these conditional probability density functions to obtain the probability density function of R given correct solution in the triangle test. Thus

$$f(R|\Delta) = \sum_{i=1}^{4} \frac{P(C_i)}{p_\Delta} f(R|C_i)$$

$$= \frac{4(3\pi\sigma^2)^{-\frac{1}{2}}}{p_\Delta} I(\sqrt{2}R/3\sigma) \cosh(2\mu R/3\sigma^2)\, e^{-(R^2+\mu^2)/3\sigma^2}, \qquad (7)$$

$0 \leq R \leq \infty$, $p_\Delta = \sum_1^4 P(C_i)$ shown in (1).

We require also the conditional distribution of W for configurations $C_5,\ldots,C_{12}$. The joint conditional distributions of x_1, x_2 and y are written down as for (2). We consider C_5 in a little detail; now the orthogonal transformation has $r = (2x_1 - x_2 - y)/\sqrt{6}$, s as before and $t = (y - x_2)/\sqrt{2}$. The insertion of W comes since $W = -\sqrt{3}\, r/\sqrt{2}$. Steps similar to those for $f(R|C_1)$ lead to

$$f(W|C_5) = \frac{(3\pi\sigma^2)^{-\frac{1}{2}}}{P(C_5)} e^{-(W-\frac{1}{2}\mu)^2/3\sigma^2}\left[I\left(\frac{\sqrt{2}W}{3\sigma} - \frac{\mu}{\sigma\sqrt{2}}\right) + I\left(\frac{\mu}{\sigma\sqrt{2}}\right)\right], \qquad (8)$$

with $0 \leq W \leq \infty$.

From symmetry, $f(W|C_6)$ has the form (8) and $f(W|C_7)$ and $f(W|C_8)$ have the form of (8) with μ replaced by $-\mu$ and simplified since $I(-a) = -I(a)$. For $f(W|C_9)$, the transformation is the same as for (8) but the boundaries of the region differ; however, the same steps give us

$$f(W|C_9) = \frac{(3\pi\sigma^2)^{-\frac{1}{2}}}{P(C_9)} e^{-(W-\frac{1}{2}\mu)^2/3\sigma^2}\left[I\left(\frac{\sqrt{2}W}{3\sigma} + \frac{\mu}{\sigma\sqrt{2}}\right) - I\left(\frac{\mu}{\sigma\sqrt{2}}\right)\right], \qquad (9)$$

$0 \leq W \leq \infty$. $f(W|C_{10})$ has the form of (9) and $f(W|C_{11})$ and $f(W|C_{12})$ have the form of (9) with μ replaced by $-\mu$.

The marginal conditional distribution of W given that the selection in the triangle test is incorrect (denoted by $\overline{\Delta}$) is available now as

$$f(W|\overline{\Delta}) = \frac{4(3\pi\sigma^2)^{-\frac{1}{2}}}{p_{\overline{\Delta}}} e^{-(W^2 + \frac{1}{4}\mu^2)/3\sigma^2} \cosh\left(\frac{\mu W}{3\sigma^2}\right)\left[I\left(\frac{\sqrt{2}W}{3\sigma} + \frac{\mu}{\sigma\sqrt{2}}\right) + I\left(\frac{\sqrt{2}W}{3\sigma} - \frac{\mu}{\sigma\sqrt{2}}\right)\right], \quad (10)$$

$0 \leq W \leq \infty$, $p_{\overline{\Delta}} = 1 - p_{\Delta}$.

V. ESTIMATION AND DIFFERENCE TESTING.

With the mathematical developments of the preceding section, we may now turn again to the problem of sensory difference testing. Suppose as before that N triangle tests yield scored differences, $R_1,\ldots,R_m$, $W_1,\ldots,W_n$, $m + n = N$ and note that this implies m correct selections of the variant sample and n incorrect selections. The likelihood function may be written as

$$L(R_1,\ldots,R_m, W_1,\ldots,W_n) = L$$

$$= p_{\Delta}^m \left[\prod_{i=1}^{m} f(R_i|\Delta)\right] p_{\overline{\Delta}}^n \left[\prod_{i=1}^{n} f(W_j|\overline{\Delta})\right].$$

Use of (7) and (10) with possible simplifications gives

$$L = 4^N (3\pi)^{-N/2} \sigma^{-N} e^{-(\sum_1^m R_i^2 + \sum_1^n W_j^2)/3\sigma^2} e^{-\mu^2(4m+n)/12\sigma^2}$$

$$\times \prod_{i=1}^{m} \cosh(2\mu R_i/3\sigma^2) \prod_{j=1}^{n} \cosh(\mu W_j/3\sigma^2)$$

$$\times \prod_{i=1}^{m} I(\sqrt{2} R_i/3\sigma) \prod_{j=1}^{n} \left[I\left(\frac{\sqrt{2}W_j}{3\sigma} + \frac{\mu}{\sigma\sqrt{2}}\right) + I\left(\frac{\sqrt{2}W_j}{3\sigma} - \frac{\mu}{\sigma\sqrt{2}}\right)\right]. \quad (11)$$

While (11) is a difficult likelihood function, we shall proceed to develop iterative procedures for maximum likelihood estimates of μ and σ and to develop the likelihood ratio test of the null hypothesis, H_o: $\mu = 0$, versus the alternative, H_a: $\mu \neq 0$.

The procedure that we have adopted is to maximize lnL with respect to $\theta = \mu/\sigma$ and σ, the reparameterization yielding minor simplifications. Then, with k independent of θ and σ,

$$\begin{aligned} \ln L = {} & k - N \ln\sigma - \Big(\sum_1^m R_i^2 + \sum_1^n W_j^2\Big)/3\sigma^2 - (4m + n)\,\theta^2/12 \\ & + \sum_1^m \ln\cosh(2\theta R_i/3\sigma) + \sum_1^n \ln\cosh(\theta W_j/3\sigma) \\ & + \sum_1^m \ln I(\sqrt{2}R_i/3\sigma) + \sum_1^n \ln\left[I\left(\frac{\sqrt{2}W_j}{3\sigma} + \frac{\theta}{\sqrt{2}}\right) + I\left(\frac{\sqrt{2}W_j}{3\sigma} - \frac{\theta}{\sqrt{2}}\right)\right]. \end{aligned} \tag{12}$$

To simplify notation, we now let

$$I_{+,j} = I\left(\frac{\sqrt{2}W_j}{3\sigma} + \frac{\theta}{\sqrt{2}}\right) \text{ and } I_{-,j} = I\left(\frac{\sqrt{2}W_j}{3\sigma} - \frac{\theta}{\sqrt{2}}\right). \tag{13}$$

Similarly, if

$$\Phi(a) = \frac{1}{\sqrt{2\pi}}\, e^{-a^2/2},$$

$$\Phi_{+,j} = \Phi\left(\frac{\sqrt{2}W_j}{3\sigma} + \frac{\theta}{\sqrt{2}}\right) \text{ and } \Phi_{-,j} = \Phi\left(\frac{\sqrt{2}W_j}{3\sigma} - \frac{\theta}{\sqrt{2}}\right). \tag{14}$$

To maximize lnL (and L) and to obtain $\hat{\theta}$ and $\hat{\sigma}$ the maximum likelihood estimators of θ and σ ($\hat{\mu} = \hat{\sigma}\,\hat{\theta}$), the partial derivatives of lnL with respect to θ and σ are set equal to zero. The resulting equations are

$$f_1(\theta,\sigma) = \sigma \frac{\partial}{\partial\theta} \ln L$$

$$= -\theta\sigma\,(4m+n)/6 + \sum_{i=1}^{m} (2R_i/3)\tanh(2\theta R_i/3\sigma)$$

$$+ \sum_{j=1}^{n} (W_j/3)\tanh(\theta W_j/3\sigma)$$

$$+ (\sigma/\sqrt{2}) \sum_{j=1}^{n} (\Phi_{+,j} - \Phi_{-,j})/(I_{+,j} + I_{-,j}) = 0 \qquad (15)$$

and

$$f_2(\theta,\sigma) = \sigma \frac{\partial}{\partial\sigma} \ln L$$

$$= -N + (2/3\sigma^2)\Big(\sum_{i=1}^{m} R_i^{\,2} + \sum_{j=1}^{n} W_j^{\,2}\Big)$$

$$- \sum_{i=1}^{m} (2\theta R_i/3\sigma)\tanh(2\theta R_i/3\sigma)$$

$$- \sum_{j=1}^{n} (\theta W_j/3\sigma)\tanh(\theta W_j/3\sigma)$$

$$- \sum_{j=1}^{m} (\sqrt{2}\,R_i/3\sigma)\,\Phi(\sqrt{2}\,R_i/3\sigma)\,/\,I(\sqrt{2}\,R_i/3\sigma)$$

$$- \sum_{j=1}^{n} (\sqrt{2}\, W_j/3\sigma)(\Phi_{+,j} + \Phi_{-,j})/(I_{+,j} + I_{-,j}) = 0. \tag{16}$$

These equations are solved iteratively (and quite easily with an electronic computer). The usual procedure is followed, namely to use initial estimates θ_o and σ_o, solve the linear equations,

$$f_1(\theta,\sigma) \simeq f_1(\theta_o, \sigma_o) + \Delta\theta \left.\frac{\partial f_1}{\partial\theta}\right|_{\theta_o,\sigma_o} + \Delta\sigma \left.\frac{\partial f_1}{\partial\sigma}\right|_{\theta_o,\sigma_o} = 0 , \tag{17}$$

and

$$f_2(\theta,\sigma) \simeq f_2(\theta_o,\sigma_o) + \Delta\theta \left.\frac{\partial f_2}{\partial\theta}\right|_{\theta_o,\sigma_o} + \Delta\sigma \left.\frac{\partial f_2}{\partial\sigma}\right|_{\theta_o,\sigma_o} = 0 , \tag{18}$$

for $\Delta\theta$ and $\Delta\sigma$, and obtain $\theta_1 = \theta_o + \Delta\theta$ and $\sigma_1 = \sigma_o + \Delta\sigma$ as improved estimates. The procedure is repeated until satisfactory convergence to $\hat{\theta}$ and $\hat{\sigma}$ is obtained. The required derivatives for (17) and (18) are:

$$\frac{\partial f_1}{\partial\theta} = - \sigma(4m + n)/6 + \sigma \sum_{i=1}^{m} (2R_i/3\sigma)^2 \operatorname{sech}^2 (2\theta R_i/3\sigma)$$

$$+ \sigma \sum_{j=1}^{n} (W_j/3\sigma)^2 \operatorname{sech}^2 (\theta W_j/3\sigma) - (\sigma/2) \sum_{j=1}^{n} (\Phi_{+,j} - \Phi_{-,j})^2/(I_{+,j} + I_{-,j})^2$$

$$+ (\sigma/2) \sum_{j=1}^{n} (\Phi'_{+,j} + \Phi'_{-,j})/(I_{+,j} + I_{-,j}),$$

$$\frac{\partial f_1}{\partial\sigma} = -\theta\ (4m+n)/6 - (\theta/\sigma^2)\sum_{i=1}^{m}(2R_i/3)^2\ \text{sech}^2\ (2\theta\ R_i/3\sigma)$$

$$-(\theta/\sigma^2)\sum_{j=1}^{n}(W_j/3)^2\ \text{sech}^2\ (\theta\ W_j/3\sigma)$$

$$+(1/\sqrt{2})\sum_{j=1}^{n}(\Phi_{+,j}-\Phi_{-,j})/(I_{+,j}+I_{-,j})$$

$$+\sum_{j=1}^{n}(W_j/3\sigma)(\Phi^2_{+,j}-\Phi^2_{-,j})/(I_{+,j}+I_{-,j})^2$$

$$+\sum_{j=1}^{n}(W_j/3\sigma)(\Phi'_{+,j}-\Phi'_{-,j})/(I_{+,j}+I_{-,j})\ ,$$

$$\frac{\partial f_2}{\partial\theta} = \frac{\partial f_1}{\partial\sigma} - f_1/\sigma\ , \qquad (19)$$

and

$$\frac{\partial f_2}{\partial\sigma} = (1/\sigma)\Big[-(4/3\sigma^2)\Big(\sum_{i=1}^{m}R_i^{\ 2}+\sum_{j=1}^{n}W_j^2\Big)+\sum_{i=1}^{m}(2\theta R_i/3\sigma)\ \tanh\ (2\theta R_i/3\sigma)$$

$$+\sum_{i=1}^{m}(2\theta R_i/3\sigma)^2\ \text{sech}^2\ (2\theta R_i/3\sigma)+\sum_{j=1}^{n}(\theta W_j/3\sigma)\ \tanh\ (\theta W_j/3\sigma)$$

$$+\sum_{j=1}^{n}(\theta W_j/3\sigma)^2\ \text{sech}^2\ (\theta W_j/3\sigma)$$

$$+\sum_{i=1}^{m}(\sqrt{2}R_i/3\sigma)\ \Phi\ (\sqrt{2}R_i/3\sigma)/I(\sqrt{2}R_i/3\sigma)$$

$$+ \sum_{i=1}^{m} (\sqrt{2}R_i/3\sigma)^2 \; \Phi'(\sqrt{2}R_i/3\sigma)/I(\sqrt{2}R_i/3\sigma)$$

$$- \sum_{i=1}^{m} (\sqrt{2}R_i/3\sigma)^2 \; \Phi^2(\sqrt{2}R_i/3\sigma)/I^2(\sqrt{2}R_i/3\sigma)$$

$$+ \sum_{j=1}^{n} (\sqrt{2}W_j/3\sigma) \; (\Phi_{+,j} + \Phi_{-,j})/(I_{+,j} + I_{-,j})$$

$$+ \sum_{j=1}^{n} (\sqrt{2}W_j/3\sigma)^2 \; (\Phi'_{+,j} + \Phi'_{-,j})/(I_{+,j} + I_{-,j})$$

$$- \sum_{j=1}^{n} (\sqrt{2}W_j/3\sigma)^2 \; (\Phi_{+,j} + \Phi_{-,j})^2 \,/\, (I_{+,j} + I_{-,j})^2 \Big] .$$

Note that if the information matrix leading to asymptotic variances and covariance of $\hat{\theta}$ and $\hat{\sigma}$ is required, this may be computed easily at the last stage of iteration since $\partial^2 \ln L/\partial\theta^2 = (\partial f_1/\partial\theta)/\sigma$, $\partial^2 \ln L/\partial\theta\partial\sigma = [(\partial f_1/\partial\sigma) - (f_1/\sigma)]/\sigma$, and $\partial^2 \ln L/\partial\sigma^2 = [(\partial f_2/\partial\sigma) - (f_2/\sigma)]/\sigma$.

Initial values θ_o and σ_o may be obtained in various ways. Examples considered have shown quite rapid convergence to $\hat{\theta}$ and $\hat{\sigma}$ even when θ_o and σ_o are quite poor first approximations. An easy procedure for obtaining θ_o is to take m/N as an estimate of $p_\triangle$ and to read $\theta_o = \mu_o/\sigma_o$ from Table I of the paper by Bradley [1963]. A satisfactory value for σ_o may be obtained by examination of the range of the R_i and W_j but an additional method is given in the appendix.

When $\hat{\theta}$ and $\hat{\sigma}$ have been obtained, the maximum of lnL is calculated.

In order to develop the required likelihood ratio test, we need also to obtain the maximum likelihood estimator of σ given that $\mu = 0$ or $\theta = 0$. Careful computer programming can include this case in the general estimation procedure or a simpler program may be developed. When $\theta = 0$, lnL in (12) is considerably simplified and the equation to be solved is $f_2(0,\sigma) = 0$. This solution is developed iteratively again from the linear equation in $\Delta\sigma$ from (18),

$$f_2(0,\sigma) \simeq f_2(0,\sigma_o) + \Delta\sigma \left.\frac{\partial f_2}{\partial\sigma}\right|_{0,\sigma_o} \tag{20}$$

with $\partial f_2/\partial\sigma$ at $(0,\sigma_o)$ obtainable from (19).

If we denote the value of σ that maximizes L given $\theta = 0$ by $\tilde{\sigma}$, the maximum of lnL is obtained through substitution of $(0,\tilde{\sigma})$ for (θ, σ) in lnL.

The usual asymptotic theory is used for the test of significance. If λ is the likelihood ratio,

$$-2 \ln \lambda = -2[\ln L|_{0,\tilde{\sigma}} - \ln L|_{\hat{\theta},\hat{\sigma}}] \tag{21}$$

is taken to have the χ^2- distribution with one degree of freedom.

VI. NUMERICAL EXAMPLES.

Data for N = 44 modified triangle tests together with scored "degree of difference" are given in Table 1. The scoring scale used is no difference : 0, very slight difference : 2, slight difference : 4, moderate difference : 6, large difference : 8, and very large difference : 10. The triangle tests compare an

TABLE 1.

SCORES ON DEGREE OF DIFFERENCE ON CEREAL TESTS[3]

R, Scores for Correct Tests		W, Scores for Incorrect Tests	
2	8	4	2
4	6	6	2
2	2	6	2
8	3	6	2
6	4	2	2
6		4	0
2		2	6
2		2	4
6		0	6
2		0	
2		2	
4		2	
8		0	
0		5	
4		4	

[3] Data provided through the courtesy of the Division, Corporation by Mr.

experimental cereal with a control cereal and some triangle tests involved two samples of the control cereal with one of the experimental and some tests used one control sample and two experimental samples. Note that $m = 20$ and $n = 24$.

To obtain $\hat{\theta}$ and $\hat{\sigma}$, the iterative procedure of equations (17) and (18) was initiated with arbitrarily selected values, $\theta_o = 1$, $\sigma_o = 2.5$. The IBM 709 computer was used with instructions to terminate iterations when successive values of θ and σ agree to the seventh decimal place. Successive values of θ, σ and lnL to four-place accuracy are given in Table 2. To obtain $\tilde{\sigma}$ given $\theta = 0$, the initial value of σ_o was again 2.5; values to four-place accuracy with values of lnL in this case also are given in Table 2. The test statistic is computed from (21),

$$- 2 \ln L \lambda = - 2 [-116.1963 - (-114.1305)] = 4.1316$$

and comparison with χ^2-tables with one degree of freedom indicates that this value comes at approximately the .044 level of signigicance.

If the usual triangle test had been used, a one-sided test of significance would have been performed with the null hypothesis being H_o: $p_\triangle = 1/3$ and the alternative being H_a: $p_\triangle > 1/3$. Using $N = 44$, $m = 20$ and the normal approximation to the binomial with continuity correction, we have

$$u = \frac{m - (N/3) \pm 1/2}{\sqrt{\frac{2N}{9}}} = \frac{3(20 - 14.6667 - .5)}{\sqrt{88}} = 1.5457$$

with significance level .061. The confidence interval on $p_\triangle$ with confidence coefficient .95 by the usual methods is (.307,602).

TABLE 2

PARAMETER VALUES AND lnL FOR SUCCESSIVE ITERATIONS

Iteration	Alternative Hypothesis			Null Hypothesis $\theta = 0$	
	θ	σ	lnL	σ	lnL
0	1	2.5	----	2.5	----
1	1.2581	1.8825	- 114.8550	2.4859	- 116.1963
2	1.1186	2.0639	- 114.1520	2.4860	- 116.1963
3	1.0820	2.1054	- 114.1306	2.4860	- 116.1963
4	1.0809	2.1069	- 114.1305	----	----
5	1.0809	2.1069	- 114.1305	----	----

A confidence interval and estimate of $p_\triangle$ may be obtained from the modified triangle test. We have our estimate of θ as 1.0809 from Table 2. Using Table I of the reference, Bradley [1963], we equate θ with μ/σ in column 1 and interpolate in column 2 to find $p_\triangle$ = .431. A confidence interval on θ may be converted into a confidence interval on $p_\triangle$. The computer program produces the required partial derivatives of f_1 and f_2 so that the matrix,

$$\begin{bmatrix} -\dfrac{\partial^2 \ln L}{\partial\theta^2} & -\dfrac{\partial^2 \ln L}{\partial\theta\partial\sigma} \\ -\dfrac{\partial^2 \ln L}{\partial\theta\partial\sigma} & -\dfrac{\partial^2 \ln L}{\partial\sigma^2} \end{bmatrix} = \begin{bmatrix} 18.4175 & 20.2982 \\ 20.2982 & 43.8007 \end{bmatrix},$$

may be easily computed. The inverse of this matrix, from large-sample theory on maximum likelihood estimators, gives the variance-covariance matrix for $\hat{\theta}$ and $\hat{\sigma}$ This matrix is

$$\begin{bmatrix} .11098 & -.05143 \\ -.05143 & .04666 \end{bmatrix}$$

so that $(\hat{\theta} - \theta)$ may be taken as normal with zero mean and variance .11098. The .95-confidence interval on θ is evaluated easily as (.4280, 1.7338) and the corresponding interval on $p_\triangle$ is (.350, .552).

Harmon [1963] used a made-up example wherein the values of x_1 and x_2 were taken from a table of random normal variates with mean zero and variance unity whereas the values of y were chosen similarly but from a population with mean 1.5 and variance unity. In his example, N = 20, m = 9 and n = 11. He found

$\hat{\theta} = 1.7097$ and $\hat{\sigma} = .9699$ under H_a and $\hat{\tilde{\sigma}} = 1.3620$ under H_o. For this example, $\chi_1^2 = -2 \ln \lambda = 4.358$ for the modified triangle test and the normal deviate $u = .870$ for the conventional binomial triangle test. The estimates of $p_\triangle$ were respectively .547 and .450 while the true value of $p_\triangle$ for the populations sampled is .5065.

The example worked by Harmon appears to give satisfactory estimates of the known parameters of the population sampled and to suggest that improved test power may be attributed to the modified triangle test. The example of Table 1 seems also to be satisfactory in that higher significance is obtained with the modified triangle test in comparison with the simple triangle test and a shorter confidence interval is obtained on $p_\triangle$. One cannot judge the merits of the modified triangle test procedure on the basis of one or two examples. Extended experience or mathematical investigations should show that the modified triangle test is more powerful than the simple triangle test in the detection of differences in sensory testing.

VII. INDETERMINATE FORMS IN COMPUTATIONS.

The theory of the preceding sections is based on a model that attributes probability zero to the occurrence of zero-values of R_i or W_j. However, in practice as in Table 1, zero-values do occur from the discreteness of scoring scales used. Unless consideration is given to this in computer programming, difficulties will occur.

If an R_i or W_j is zero, indeterminancies occur in f_1, f_2, $\partial f_1/\partial\theta$, $\partial f_1/\partial\sigma$, $\partial f_2/\partial\theta$ and $\partial f_2/\partial\sigma$. Programming should be done so that, if R_i or W_j is zero, the appropriate limits replace the indeterminate forms. The necessary limits are as follows:

$$\lim_{x\to 0} x\Phi(x)/I(x) = 1, \tag{22}$$

$$\lim_{x\to 0} x[\Phi_+ + \Phi_-]/[I_+ + I_-] = 1, \tag{23}$$

$$\lim_{x\to 0} [\Phi_+ - \Phi_-]/[I_+ + I_-] = -\theta/\sqrt{2}, \tag{24}$$

$$\lim_{x\to 0} [\Phi_+ - \Phi_-]^2/[I_+ + I_-]^2 = -\theta^2/2, \tag{25}$$

$$\lim_{x\to 0} [\Phi_+' + \Phi_-']/[I_+ + I_-] = \frac{1}{2}\theta^2 - 1, \tag{26}$$

$$\lim_{x\to 0} x[\Phi_+^2 - \Phi_-^2]/[I_+ + I_-]^2 = -\theta/\sqrt{2}, \tag{27}$$

$$\lim_{x\to 0} x[\Phi_+' - \Phi_-']/[I_+ + I_-] = -\theta/\sqrt{2}, \tag{28}$$

$$\lim_{x\to 0} x^2\, \Phi'(x)/I(x) = 0, \tag{29}$$

$$\lim_{x\to 0} x^2\, \Phi^2(x)/I^2(x) = 1, \tag{30}$$

$$\lim_{x\to 0} x^2[\Phi_+' + \Phi_-']/[I_+ + I_-] = 0, \tag{31}$$

and

$$\lim_{x\to 0} x^2[\Phi_+ + \Phi_-]^2/[I_+ + I_-]^2 = 1. \tag{32}$$

Rules must also be developed in computing the maximum of lnL. Subroutines for computers for logarithms usually write ln0 = 0 and this is not satisfactory.

Consider lnL in (12) and note that to calculate $-2\ln\lambda$ in (21), some rules for

$$\lim_{x \to 0} [\ln I(x/\hat{\sigma}) - \ln I(x/\tilde{\sigma})] = \ln \tilde{\sigma} - \ln \hat{\sigma}$$

and

$$\lim_{x \to 0} \{\ln [I(\frac{x}{\hat{\sigma}} + \frac{\theta}{\sqrt{2}}) + I(\frac{x}{\hat{\sigma}} - \frac{\theta}{\sqrt{2}})] - \ln 2 I(x/\tilde{\sigma})\}$$

$$= \ln\tilde{\sigma} - \ln\hat{\sigma} - \frac{1}{4}\theta^2$$

must be devised. It is satisfactory to program the computer so that

$$\ln I(x/\sigma)\Big|_{x=0} = -\ln\sigma \tag{33}$$

and

$$\ln \left[I\left(\frac{x}{\sigma} + \frac{\theta}{\sqrt{2}}\right) + I\left(\frac{x}{\sigma} - \frac{\theta}{\sqrt{2}}\right)\right]\Bigg|_{x=0} = -\ln\sigma - \frac{1}{4}\theta^2. \tag{34}$$

These rules seem to be the simplest workable ones and yield the proper value of $-2 \ln \lambda$; note that if zero-values of R_i and W_j are replaced by a small number, say .001, different values of lnL will be obtained than with the above rules but approximately the same value of $-2 \ln\lambda$ will result. These rules have been used in the first example of the preceding section.

VIII. DISCUSSION.

The procedures for the modified triangle test hold the possibility of improved efficiency in sensory difference testing. It is true that numerical work is greatly increased in the analysis in comparison with the simple triangle test; however, electronic computers are now widely available and once a computer program is developed, application of the modified triangle test is easy. The example given in this paper is shown in sufficient detail to check on programming and may be used for this purpose.

Further investigation of the modified triangle test could be useful. Consideration of power, either by asymptotic theory for large N or by Monte Carlo techniques, would be desirable. The effects of discreteness in degree-of-difference scores in comparison with the model used should not be serious but might be investigated. Routine use of modified triangle tests should lead to experience in their merits relative to simple triangle tests and such experience should be reported in subject matter literature.

Bradley [1963] shows the association between triangle tests and duo-trio tests. A modified duo-trio test could be developed in just the same way as for the method of this paper for the modified triangle test but evidence to date suggests that the triangle test should be used in preference to the duo-trio test.

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APPENDIX

MOMENTS OF R AND W

We have considered first and second moments of R and W and they yield very cumbersome general expressions. However, when μ (or θ) is zero, these expressions reduce and permit an estimate of σ that may be used in the iterative procedures developed for maximum likelihood estimation as an initial trial value.

We obtain $E(R|\triangle, \mu = 0)$, $E(W|\bar{\triangle}, \mu = 0)$, $E(R^2|\triangle, \mu = 0)$, and $E(W^2|\bar{\triangle}, \mu = 0)$ by straight-forward integration after polar transformation using (7) and (10):

$$2p_{\triangle}E(R|\triangle, \mu = 0) = p_{\bar{\triangle}}E(W|\bar{\triangle}, \mu = 0)$$

$$= \frac{8}{\sqrt{3\pi}\sigma} \int_0^\infty R\, I\left(\frac{\sqrt{2}R}{3\sigma}\right) e^{-R^2/3\sigma^2} dR$$

$$= \sqrt{3}\,\sigma / \sqrt{\pi}\ , \qquad \text{(A1)}$$

$$2p_{\triangle}E(R^2|\triangle, \mu = 0) = p_{\bar{\triangle}}E(W^2|\bar{\triangle}, \mu = 0)$$

$$= \frac{8}{\sqrt{3\pi}\sigma} \int_0^\infty R^2 I\left(\frac{\sqrt{2}R}{3\sigma}\right) e^{-R^2/3\sigma^2} dR$$

$$= \sigma^2\,[1 + (3\sqrt{3}/2\pi)]. \qquad \text{(A2)}$$

Let D be the unconditional random variable representing degree of difference so that D = R when the triangle test is correct and D = W when the triangle test is incorrect. Then

$$E(D|\ \mu = 0) = p_{\triangle}E(R|\triangle,\ \mu = 0) + p_{\bar{\triangle}}E(W|\bar{\triangle},\ \mu = 0)$$

$$= 3\sqrt{3}\ \sigma/2\ \sqrt{\pi} \tag{A3}$$

from (A1). Similarly,

$$E(D^2|\mu = 0) = (3\ \sigma^2/2) + (9\sqrt{3}\sigma^2/\ 4\pi) \tag{A4}$$

from (A2). The variance of D, given $\mu = 0$, in the usual way is

$$V(D|\mu = 0) = (3\sigma^2/2) - (9\sqrt{3}\sigma^2/4\pi)(\sqrt{3} - 1) = .5919\sigma^2.$$

An initial value of σ may be obtained by computation of the sample variance of $R_1,\ldots,R_m,\ W_1,\ldots,W_n$ with these observations taken as a single sample of size $m+n$. If this sample variance is s^2, equate s^2 to $V(D|\mu = 0)$ and obtain

$$\sigma_o = s\ /\ \sqrt{.5919}\ . \tag{A5}$$

It appears that σ_o obtained in this way should be a satisfactory initial estimate of σ for use either in (17) and (18) or in (20). The maximum likelihood estimates of σ do not appear to differ greatly whether or not θ is taken to be zero in the iterative processes and in general $\tilde{\sigma}$ is only a little greater than $\hat{\sigma}$.